Constructive enumeration and uniform random sampling of DAGs

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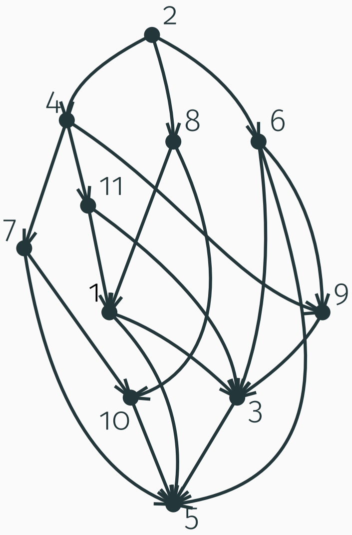
Outline

Directed Acyclic Graphs

- A finite set of vertices V e.g. $\{1, 2, \ldots, n\}$;
- a set of directed edges $E \subseteq V \times V$;
- no cycles: $v_1 \to v_2 \to \cdots \to v_n = v_1$.



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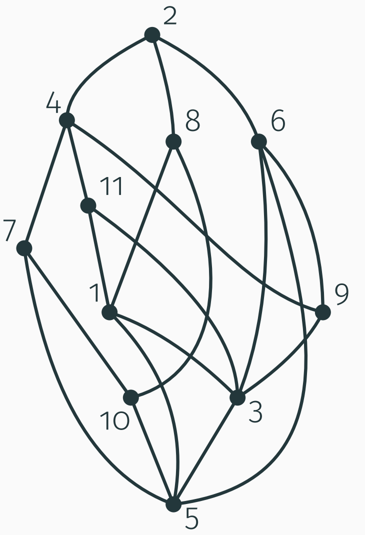

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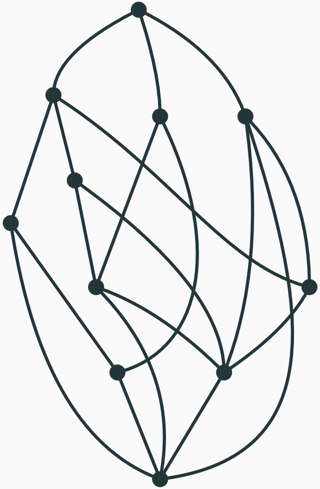

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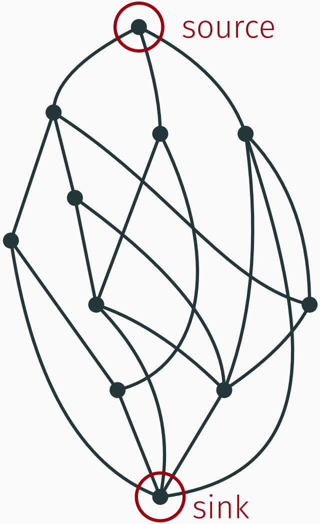

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Asymptotics

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Problems:

- ● Inclusion-exclusion
- ● No or little control over the number of edges

Still missing

- Finer control over the number of edges?
- Sampling of unlabelled structures?

Outline

Background

Directed Ordered Acyclic Graphs

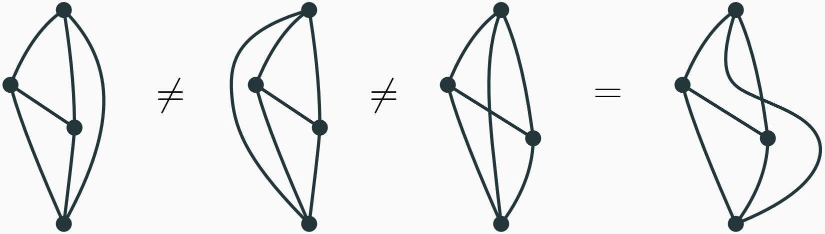
Extensions

Conclusion

A new kind of DAG

Directed Ordered Acyclic Graphs (DOAGs)

DOAG = Unlabelled DAG
+ a total order on the **outgoing** edges of each vertex
+ only one sink and one source



Motivation

- Real-life implementations of DAGs have an **ordering**; →

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  int                out_degree;
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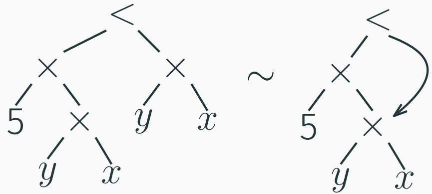

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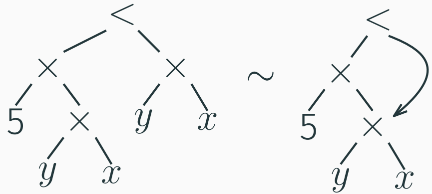

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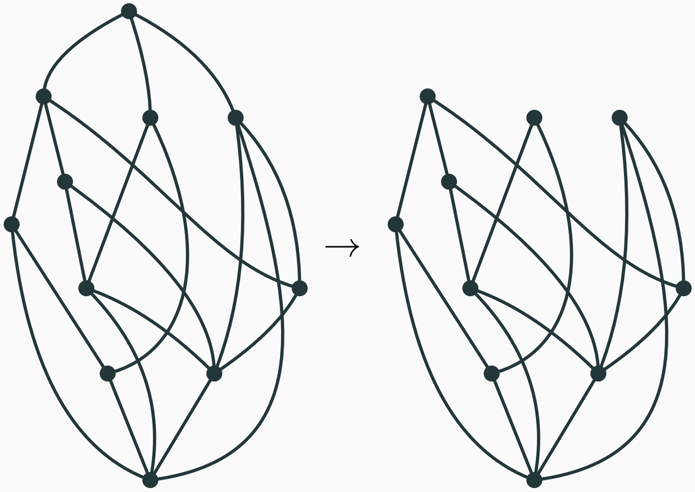
Recursive decomposition: multi-source DOAGs

Idea: remove the source and see what is left.



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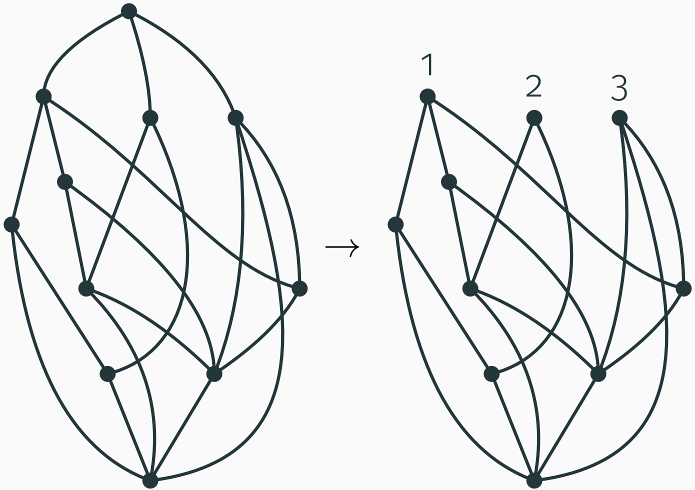

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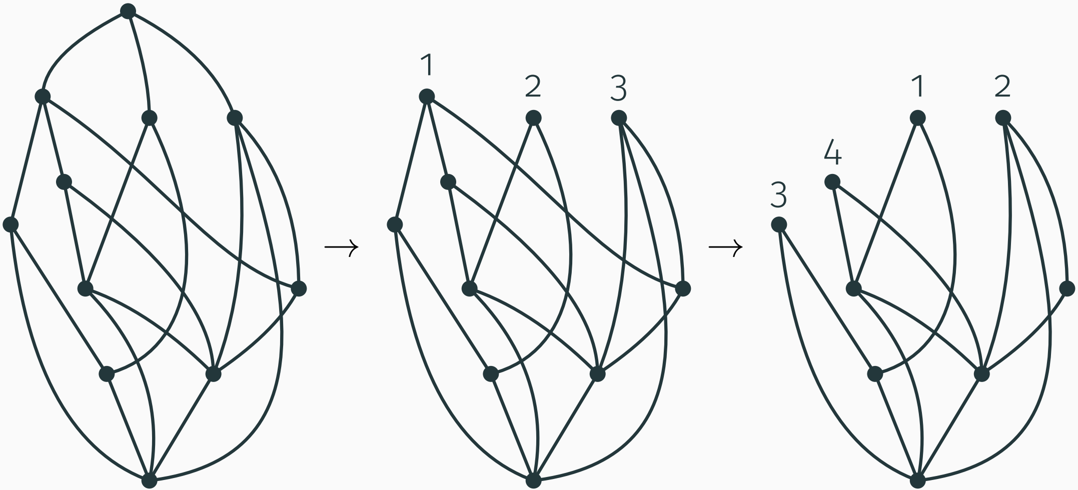

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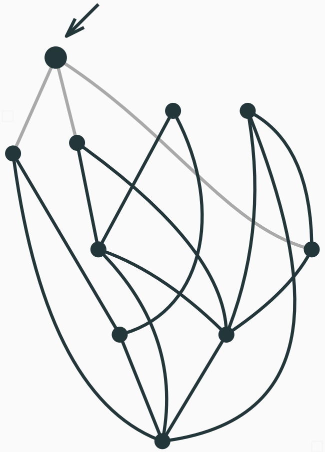

The smallest source + a sub-DOAG;

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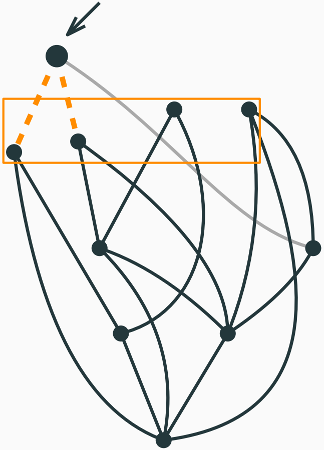

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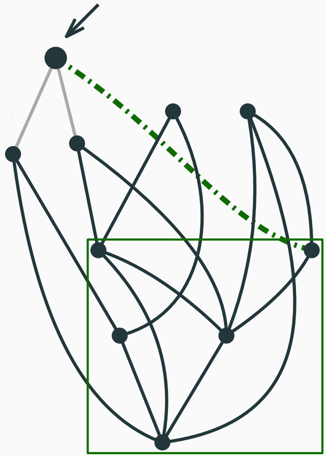

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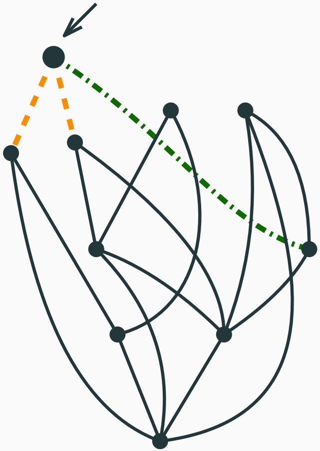

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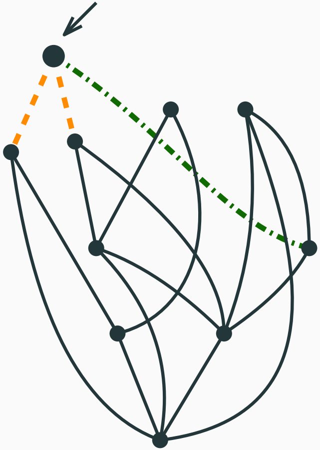

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$$D_{n,m,k} = \#\text{DOAGs with } n \text{ vertices, } m \text{ edges, } k \text{ sources}$$
$$= \sum_{s+q>0} D_{n-1,m-s-q,k-1+q} \binom{n-k-q}{s} \binom{s+q}{q} s!$$

Complexity of the counting

$$D_{1,m,k} = \mathbb{1}_{\{m=0 \wedge k=1\}}$$

$$D_{n,m,k} = 0 \qquad \text{when } k \leq 0$$

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Computing $D_{n,m,k}$ for all $n, k \leq N$ and $m \leq M$ takes $O(N^4 M)$ arithmetic operations.

In practice we reach $M = 400$, $N = M + 1$.

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Do the same, but backwards!

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1. Select (s, q) with probability $\frac{D_{n-1,m-s-q,k-1+q}\binom{n-k-q}{s}\binom{s+q}{q}s!}{D_{n,m,k}}$;

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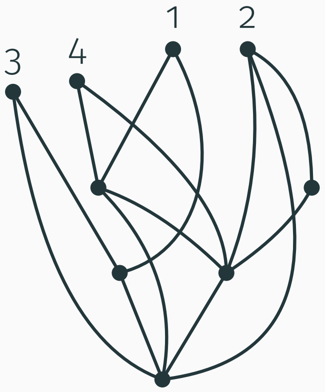

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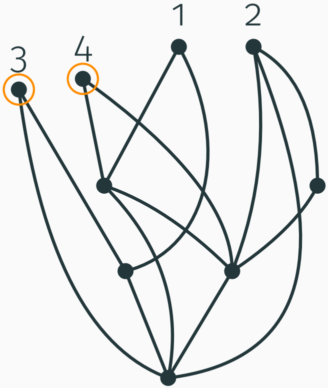

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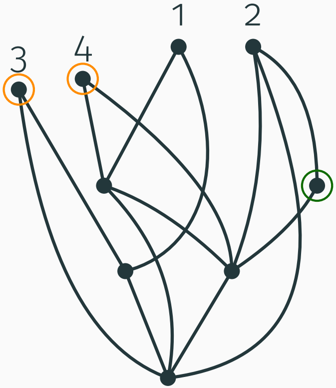

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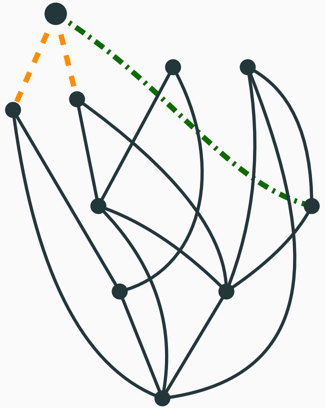

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4. Choose s internal vertices;
5. Connect them to the new sources.

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where d_v = out-degree of v.

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$$\#\text{ arithmetic operations} = O\left(\sum_{v\text{ vertex}} d_v^2\right)$$

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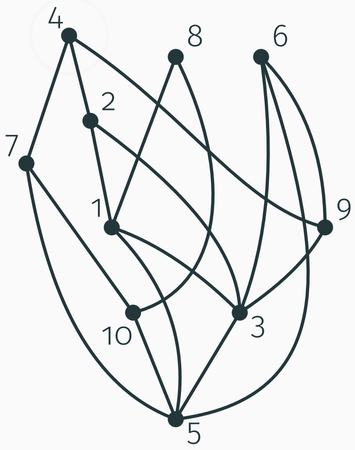
In practice: a few milliseconds.

Outline

What about labelled DAGs?

Idea: mark one source, and remove it.



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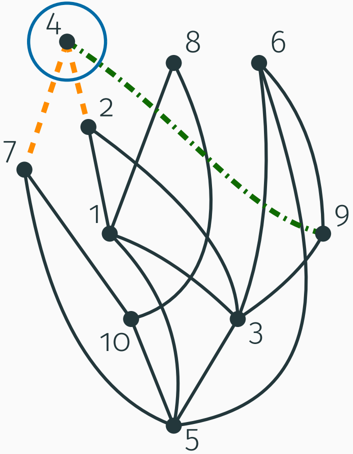

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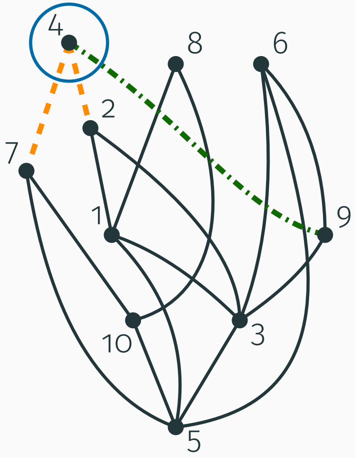

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$$n \cdot \sum_{s+q>0} V_{n-1,m-s-q,k-1+q} \binom{k-1+q}{q} \binom{n-q-k}{s}$$

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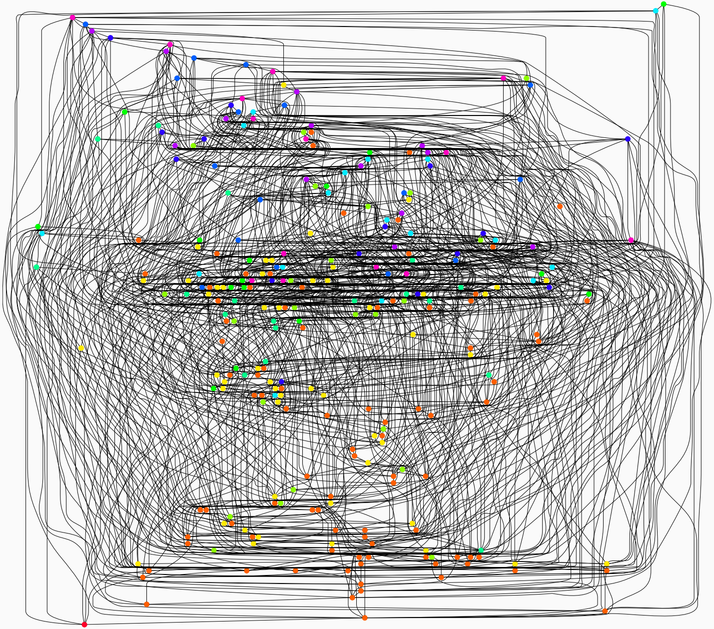
- Counting: $O(N^2 d^4)$
- Sampling: $O(N d^2)$
- In practice we reached $m = 1500$ with $d = 2$ and $m = 1000$ with $d = 10$.

Your next favourite wallpaper



Uniform DOAG with $m = 1000$ and $d = 10$.

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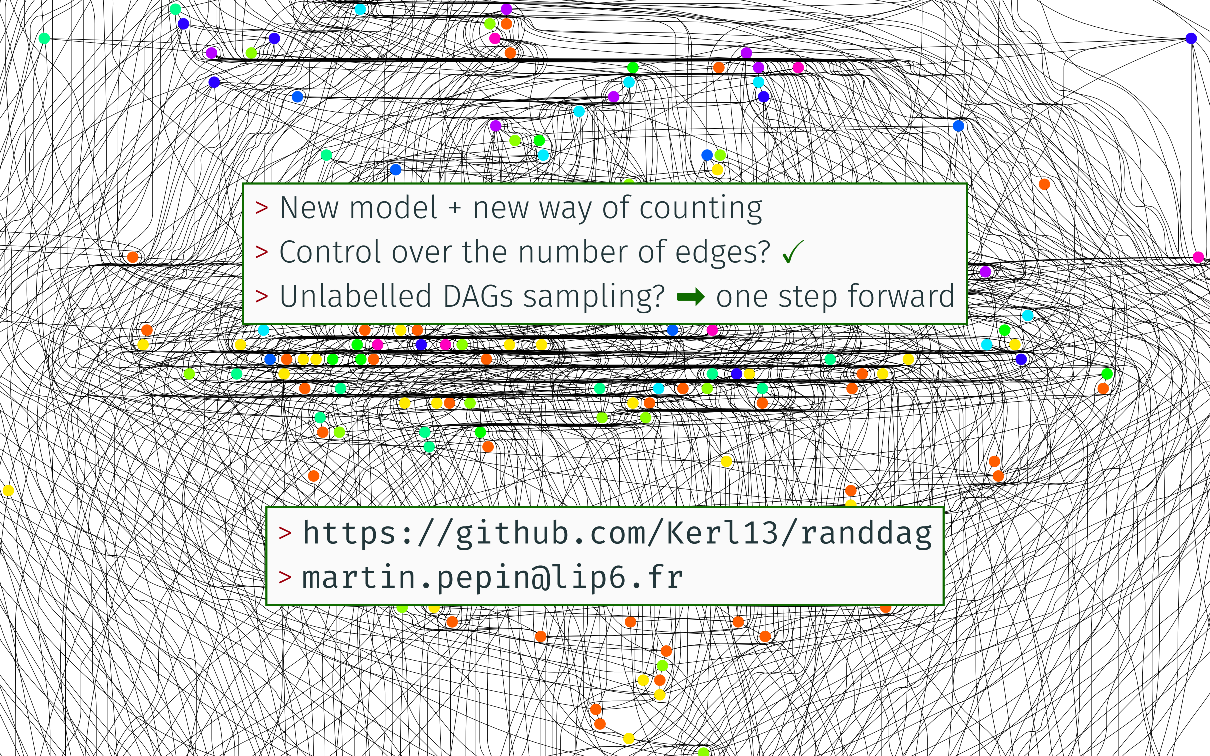

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- https://github.com/Kerl13/randdag
- martin.pepin@lip6.fr

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