# Constructive enumeration and uniform random sampling of DAGs

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Work submitted to the LAGOS conference

December 10, 2020

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# Outline

Background

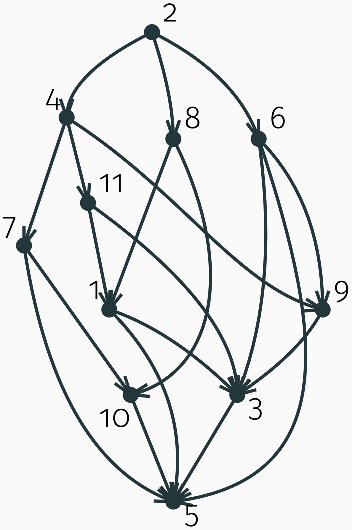
Directed Ordered Acyclic Graphs

Extensions

Conclusion and future work

# Directed Acyclic Graphs

- A finite set of vertices $V$ e.g. $\{1, 2, \ldots, n\}$;
- a set of directed edges $E \subseteq V \times V$;
- no cycles: $v_1 \to v_2 \to \cdots \to v_n = v_1$.



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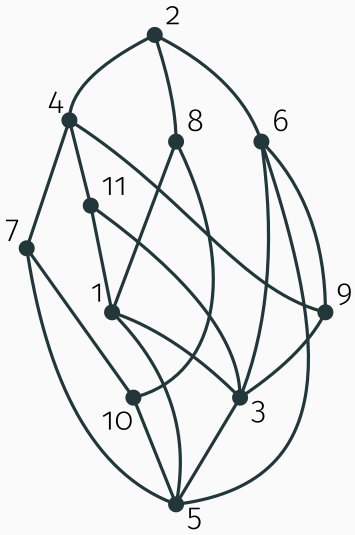

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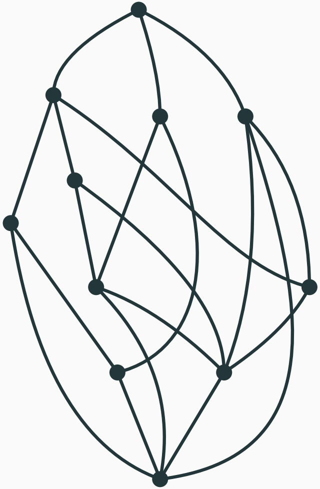

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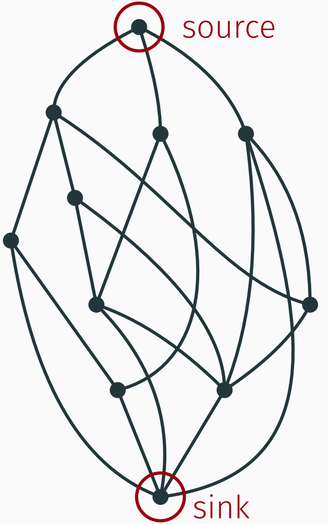

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**Problems:**

- Inclusion-exclusion
- No or little control over the number of edges

# Still missing

- Finer control over the number of edges?
- Sampling of unlabelled structures?

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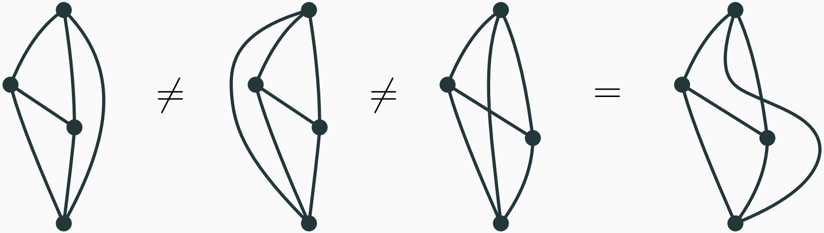
# A new kind of DAG

**Directed Ordered Acyclic Graphs (DOAGs)**

DOAG = Unlabelled DAG
+ a total order on the **outgoing** edges of each vertex
+ only one sink and one source



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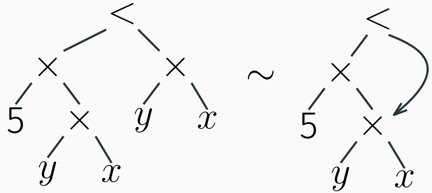

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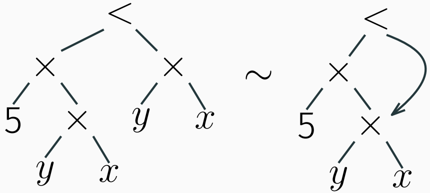
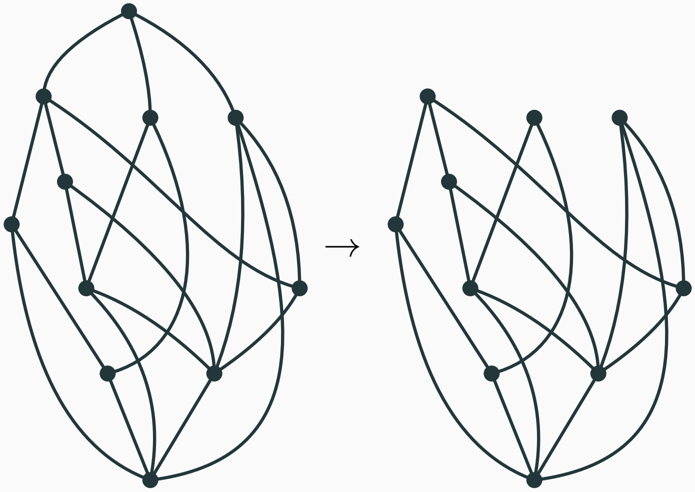


> Models **unlabelled** objects.

# Recursive decomposition: multi-source DOAGs

Idea: remove the source and see what is left.



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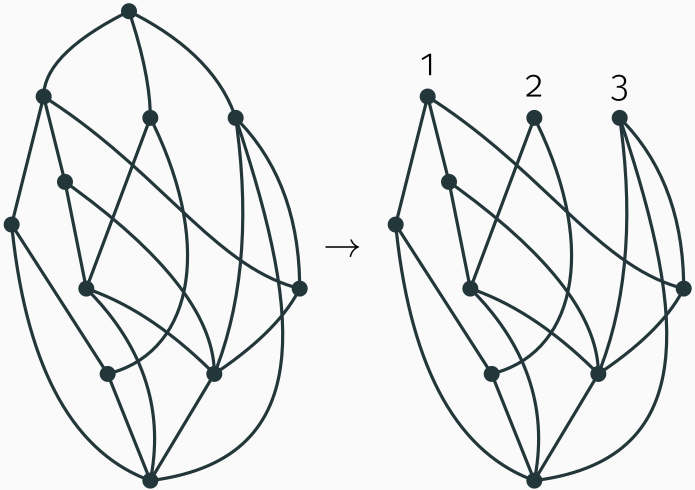

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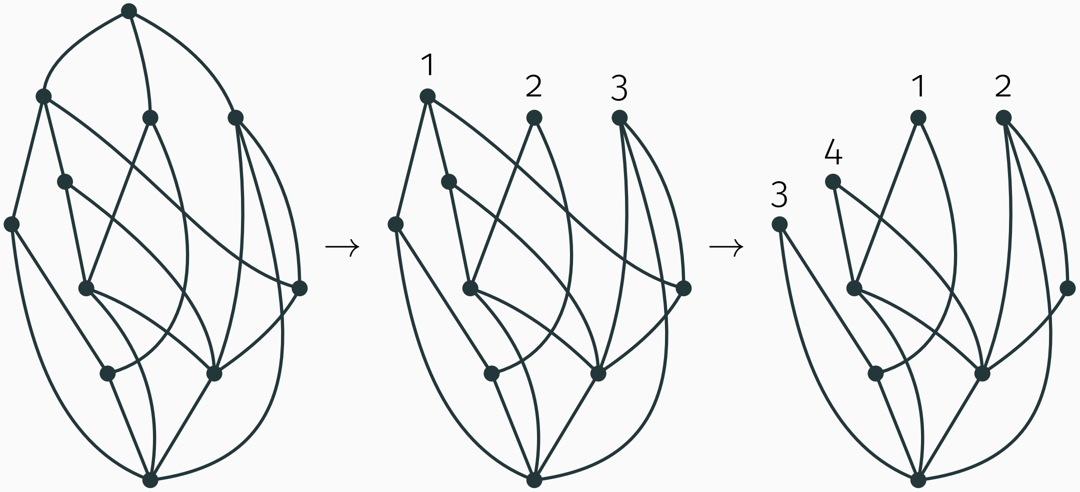

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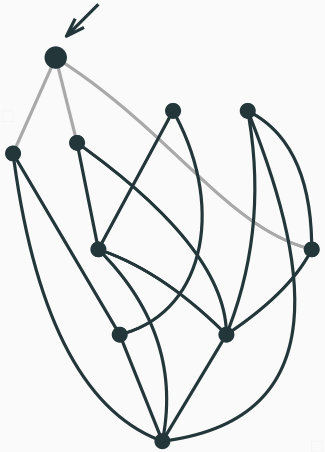

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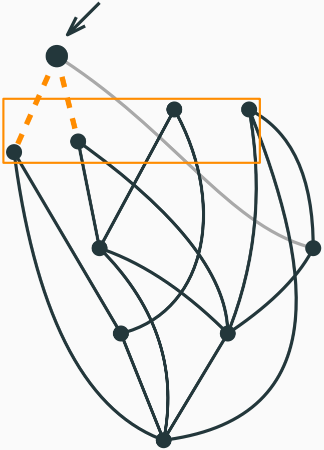

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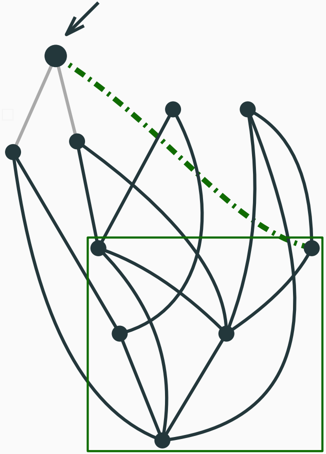

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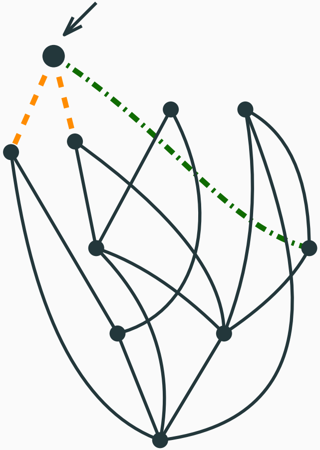

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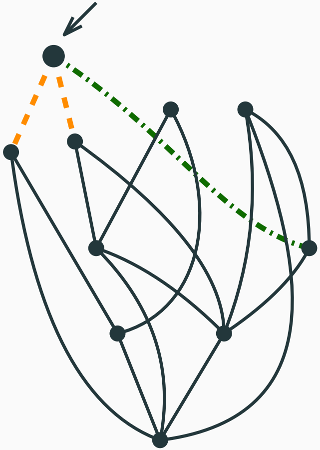

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$$D_{n,m,k} = \#\text{DOAGs with } n \text{ vertices, } m \text{ edges, } k \text{ sources}$$

$$= \sum_{s+q>0} D_{n-1,m-s-q,k-1+q} \binom{s+q}{s} \binom{n-k-q}{s} s!$$

# Complexity of the counting

$$\begin{aligned} D_{1,m,k} &= \mathbb{1}_{\{m=0 \wedge k=1\}} && \\ D_{n,m,k} &= 0 && \text{when } k \leq 0 \\ D_{n,m,k} &= \sum_{s+q>0} D_{n-1,m-s-q,k-1+q} \binom{s+q}{s} \binom{n-k-q}{s} s! && \text{when } n > 1 \end{aligned}$$

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Computing $D_{n,m,k}$ for all $n, k \leq N$ and $m \leq M$ takes $O(N^4 M)$ arithmetic operations.

In practice we reach $M = 400$, $N = M + 1$.

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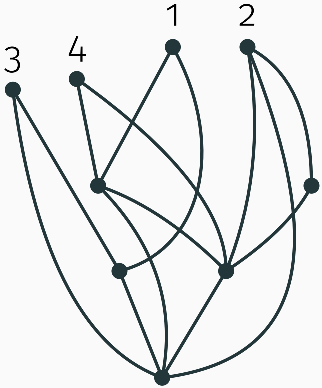


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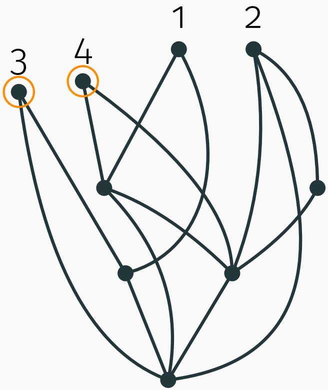


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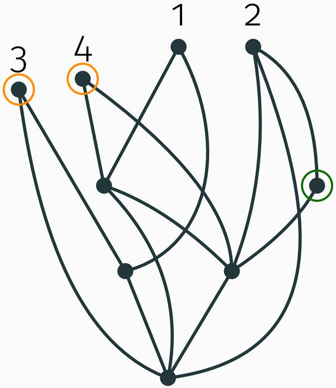


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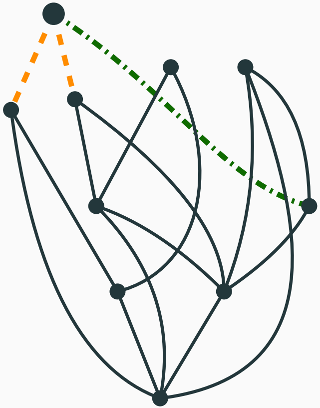

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4. Choose $s$ internal vertices;
5. Connect them to the new sources.

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4. (bonus) sum in the lexicographic order for $(s+q, s)$.

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In practice it takes a few milliseconds.

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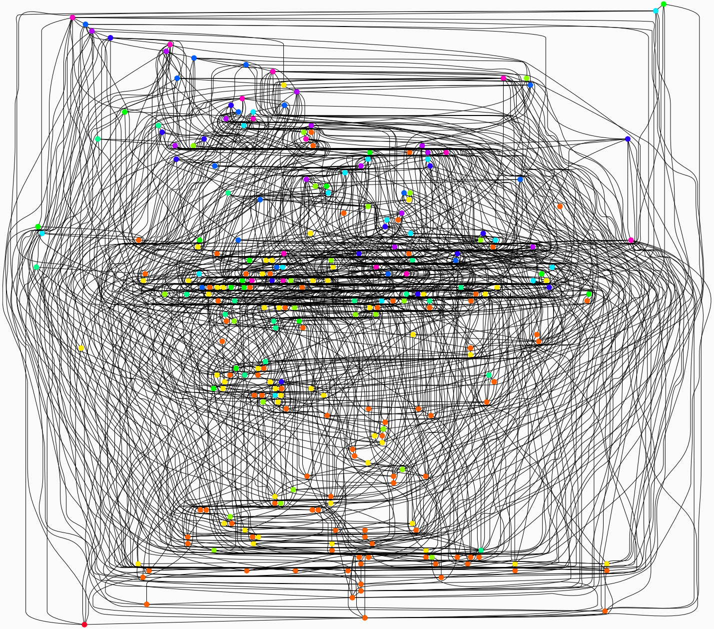
- Counting: $O(N^2 d^4)$ arithmetic operations.
- Sampling $O(N d^2)$ arithmetic operations.
- In practice we reached $m = 1500$ with $d = 2$ and $m = 1000$ with $d = 10$.

# Your next favourite wallpaper



Uniform DOAG with $m = 1000$ edges and with out-degree bounded by $d = 10$.

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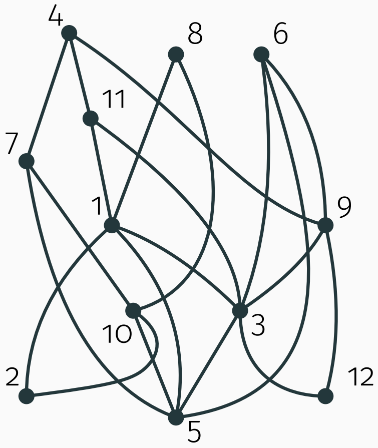

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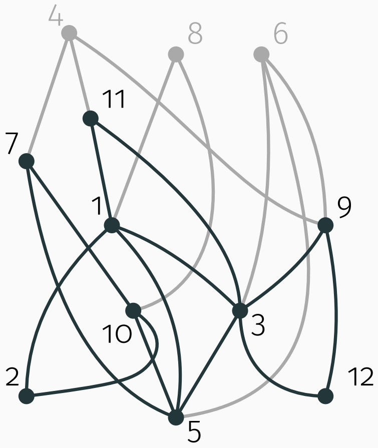

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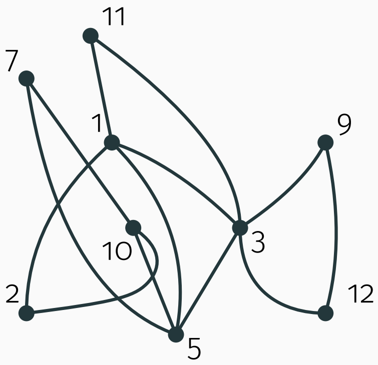

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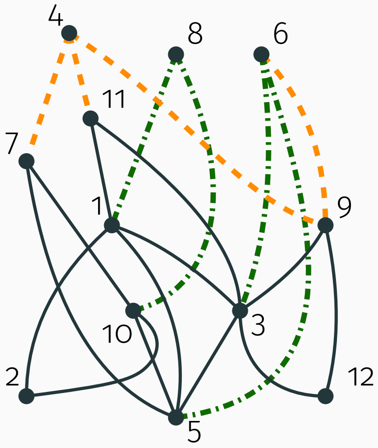


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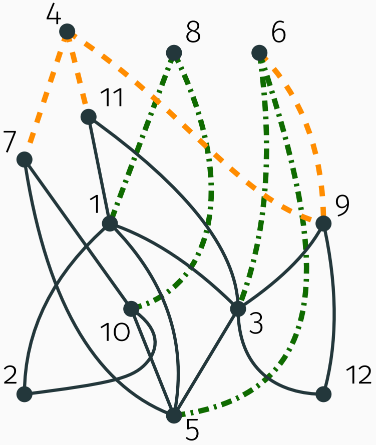


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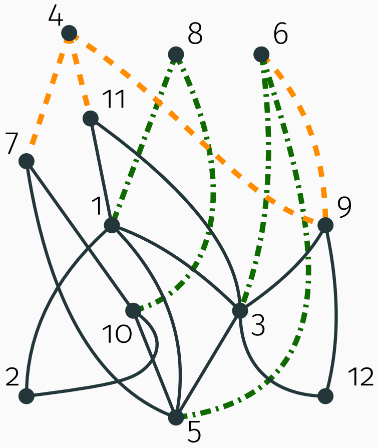


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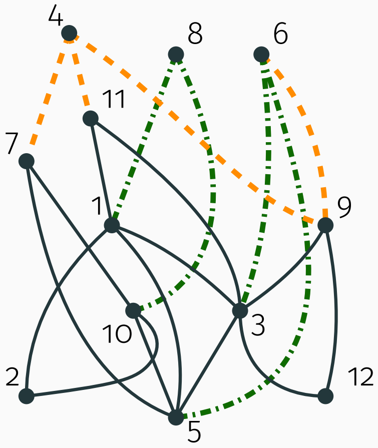


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→ Use our approach!

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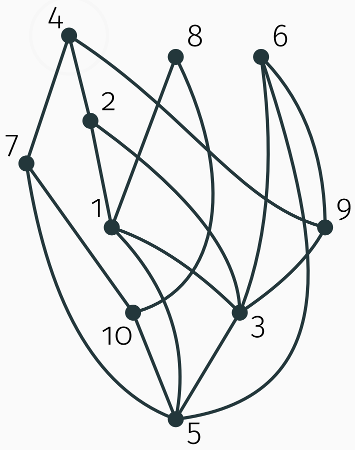


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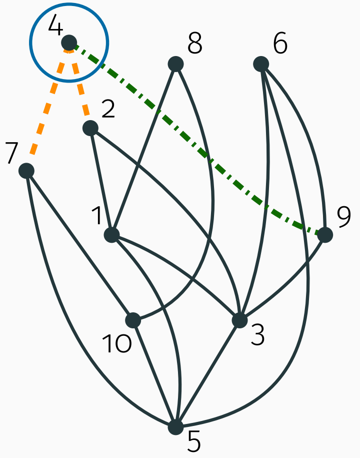


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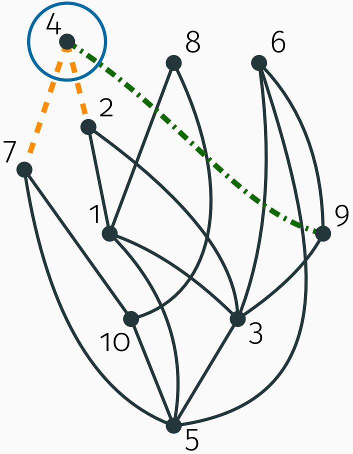


$A_{n,m,k}$ = #DAGs (one sink, $k$ sources)

$$k \cdot A_{n,m,k} =$$

$$n \cdot \sum_{s+q>0} A_{n-1,m-s-q,k-1+q} \binom{k-1+q}{q} \binom{n-q-k}{s}$$

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# Conclusion

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We presented:

- a new model of DAGs: DOAGs;
- a new way of counting.

# Future work

- Can we get rid of the one-sink-one-source constraint while retaining weak connectivity?
- Is there a symbolic method operator hidden behind the vertex-by-vertex decomposition?
- Asymptotics?
- Can we get closer to sampling regular unlabelled DAGs?

Thank you for your attention!

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